Lecture 4: Perceptrons and Multilayer Perceptrons

Cognitive Systems II - Machine Learning

Part I: Basic Approaches of Concept Learning

Perceptrons, Artificial Neuronal Networks (ANNs)

last change November 16, 2009
Biological Motivation

Biological learning systems are built of complex webs of interconnected neurons.

**motivation:**
- capture kind of highly parallel computation
- based on distributed representation

**goal:**
- obtain highly effective machine learning algorithms, independent of whether these algorithms fit biological processes (*no cognitive modeling!*)

Lecture 4: Perceptrons and Multilayer Perceptrons – p. 2
## Biological Motivation

<table>
<thead>
<tr>
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<th>Computer</th>
<th>Brain</th>
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<tbody>
<tr>
<td>computation units</td>
<td>1 CPU (&gt; $10^7$ Gates)</td>
<td>$10^{11}$ neurons</td>
</tr>
<tr>
<td>memory units</td>
<td>512 MB RAM</td>
<td>$10^{11}$ neurons</td>
</tr>
<tr>
<td>clock</td>
<td>$500$ GB HDD</td>
<td>$10^{14}$ synapses</td>
</tr>
<tr>
<td>transmission</td>
<td>$10^{-8}$ sec</td>
<td>$10^{-3}$ sec</td>
</tr>
<tr>
<td></td>
<td>$&gt; 10^9$ bits/sec</td>
<td>$&gt; 10^{14}$ bits/sec</td>
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- **Computer:** serial, quick
- **Brain:** parallel, slowly, robust to noisy data
Appropriate Problems

**BACKPROPAGATION** algorithm is the most commonly used ANN learning technique with the following characteristics:

- instances are represented as many attribute-value pairs
  - input values can be any real values
- target function output may be discrete-, real- or vector-valued
- training examples may contain errors
- long training times are acceptable
- fast evaluation of the learned target function may be required
  - many iterations may be neccessary to converge to a good approximation
- ability of humans to understand the learned target function is not important
  - learned weights are not intuitively understandable
Perceptrons

takes a vector of real-valued inputs \((x_1, \ldots, x_n)\) weighted with \((w_1, \ldots, w_n)\)

calculates the linear combination of these inputs

\[
\sum_{i=0}^{n} w_i x_i = w_0 x_0 + w_1 x_1 + \ldots + w_n x_n
\]

\(-w_0\) denotes a threshold value, i.e. that value which must be reached by the linear combination of inputs to cause the perceptron to output 1

\(x_0\) is always 1

outputs 1 if the result is greater than 0, otherwise -1
Representational Power

- A perceptron represents a hyperplane decision surface in the $n$-dimensional space of instances.
- Some sets of examples cannot be separated by any hyperplane, those that can be separated are called linearly separable.
- Many Boolean functions can be represented by a perceptron: AND, OR, NAND, NOR.
Perceptron Training Rule

**problem:** determine a weight vector $\vec{w}$ that causes the perceptron to produce the correct output for each training example

**perceptron training rule:**

$$w_i = w_i + \Delta w_i \text{ where } \Delta w_i = \eta(t - o)x_i$$

- $t$ target output
- $o$ perceptron output
- $\eta$ learning rate (usually some small value, e.g. 0.1)

**algorithm:**

1. initialize $\vec{w}$ to random weights
2. repeat, until each training example is classified correctly
   (a) apply perceptron training rule to each training example

convergence guaranteed provided linearly separable training examples and sufficiently small $\eta$
Delta Rule

- perceptron rule fails if data is not linearly separable
- delta rule converges toward a best-fit approximation
- uses gradient descent to search the hypothesis space
  - perceptron cannot be used, because it is not differentiable
  - hence, a unthresholded linear unit is appropriate
  - error measure: \( E(\vec{w}) \equiv \frac{1}{2} \sum_{d \in D} (t_d - o_d)^2 \)
- to understand gradient descent, it is helpful to visualize the entire hypothesis space with
  - all possible weight vectors and
  - associated \( E \) values
the axes $w_0, w_1$ represent possible values for the two weights of a simple linear unit

⇒ error surface must be **parabolic with a single global minimum**
Derivation of Gradient Descent

**problem:** How calculate the steepest descent along the error surface?

- derivative of $E$ with respect to each component of $\vec{w}$
  - this vector derivative is called *gradient* of $E$, written $\nabla E(\vec{w})$
    
    $$\nabla E(\vec{w}) \equiv \left[ \frac{\partial E}{\partial w_0}, \frac{\partial E}{\partial w_1}, \ldots, \frac{\partial E}{\partial w_n} \right]$$

- $\nabla E(\vec{w})$ specifies the steepest ascent, so $-\nabla E(\vec{w})$ specifies the steepest descent

**training rule:** $w_i = w_i + \Delta w_i$

$$\Delta w_i = -\eta \frac{\partial E}{\partial w_i} \quad \text{and} \quad \frac{\partial E}{\partial w_i} = \sum_{d \in D} (t_d - o_d)(-x_{id})$$

$$\Rightarrow \Delta w_i = \eta \sum_{d \in D} (t_d - o_d)x_{id}$$
Differentiating E

\[
\frac{\partial E}{\partial w_i} =
\]

\[
\frac{\partial}{\partial w_i} \frac{1}{2} \sum_{d \in D} (t_d - o_d)^2 =
\]

\[
\frac{1}{2} \sum_{d \in D} \frac{\partial}{\partial w_i} (t_d - o_d)^2 =
\]

\[
\frac{1}{2} \sum_{d \in D} 2(t_d - o_d) \frac{\partial}{\partial w_i} (t_d - o_d) =
\]

\[
\sum_{d \in D} (t_d - o_d) \frac{\partial}{\partial w_i} (t_d - \vec{w} \vec{x}_d) =
\]

\[
\sum_{d \in D} (t_d - o_d)(-x_{id})
\]

Remember:

Outer and inner derivation for \( y = u^2 \): \( \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \)

with \( u = t_d - o_d \)
Incremental Gradient Descent

- application difficulties of gradient descent
  - convergence may be quite slow
  - in case of many local minima, the global minimum may not be found

- idea: approximate gradient descent search by updating weights incrementally, following the calculation of the error for each individual example

\[ \Delta w_i = \eta(t - o)x_i \text{ where } E_d(\bar{w}) = \frac{1}{2}(t_d - o_d)^2 \]

- key differences:
  - weights are not summed up over all examples before updating
  - requires less computation
  - better for avoidance of local minima
Gradient Descent Algorithm

GRADIENT-DESCENT($training\_examples, \eta$)

Each training example is a pair of the form $< \vec{x}, t >$, where $\vec{x}$ is the vector of input values, and $t$ is the target output value. $\eta$ is the learning rate.

- Initialize each $w_i$ to some small random value
- Until the termination condition is met, Do
  - Initialize each $\Delta w_i$ to zero
  - For each $< \vec{x}, t >$ in $training\_examples$, Do
    - Input the instance $\vec{x}$ to the unit and compute the output $o$
    - For each linear unit weight $w_i$, Do $\Delta w_i = \Delta w_i + \eta(t - o)x_i$*
    - For each linear unit weight $w_i$, Do $w_i \leftarrow w_i + \Delta w_i^{**}$

To implement incremental approximation, equation ** is deleted and equation * is replaced by $w_i \leftarrow w_i + \eta(t - o)x_i$. 

Lecture 4: Perceptrons and Multilayer Perceptrons – p. 13
Perceptron vs. Delta Rule

**perceptron training rule:**
- uses thresholded unit
- converges after a finite number of iterations
- output hypothesis classifies training data perfectly
- linearly separability neccessary

**delta rule:**
- uses unthresholded linear unit
- converges asymptotically toward a minimum error hypothesis
- termination is not guaranteed
- linear separability not neccessary
Multilayer Networks (ANNs)

- capable of learning **nonlinear decision surfaces**
- normally **directed** and **acyclic** ⇒ Feed-forward Network
- based on **sigmoid unit**
  - much like a perceptron
  - but based on a smoothed, **differentiable threshold function**

\[
\sigma(\text{net}) = \frac{1}{1 + e^{-\text{net}}}
\]

\[
\lim_{\text{net} \to +\infty} \sigma(\text{net}) = 1
\]

\[
\lim_{\text{net} \to -\infty} \sigma(\text{net}) = 0
\]
BACKPROPAGATION

- learns weights for a feed-forward multilayer network with a fixed set of neurons and interconnections

- employs gradient descent to minimize error

- redefinition of $E$
  - has to sum the errors over all units
  - $E(\vec{w}) \equiv \frac{1}{2} \sum_{d \in D} \sum_{k \in \text{outputs}} (t_{kd} - o_{kd})^2$

- **problem**: search through a large $H$ defined over all possible weight values for all units in the network
BACKPROPAGATION algorithm

\[ \text{BACKPROPAGATION}(training\_examples, \eta, n_{in}, n_{out}, n_{hidden}) \]

The input from unit \( i \) to unit \( j \) is denoted \( x_{ji} \) and the weight from unit \( i \) to unit \( j \) is denoted \( w_{ji} \).

- create a feed-forward network with \( n_{in} \) inputs, \( n_{hidden} \) hidden units, and \( n_{out} \) output units
- Initialize all network weights to small random numbers
- Until the termination condition is met, Do (EPOCHE)
  - For each \( \langle \vec{x}, \vec{t} \rangle \) in \( training\_examples \), Do
    - Propagate the input forward through the network:
      1. Input \( \vec{x} \) to the network and compute \( o_u \) of every unit \( u \)
    - Propagate the errors back through the network:
      2. For each network output unit \( k \), calculate its error term \( \delta_k \)
         \[ \delta_k \leftarrow o_k(1 - o_k)(t_k - o_k) \]
      3. For each hidden unit \( h \), calculate its error term \( \delta_h \)
         \[ \delta_h \leftarrow o_h(1 - o_h) \sum_{k \in \text{outputs}} w_{kh} \delta_k \]
      4. Update each weight \( w_{ji} \)
         \[ w_{ji} \leftarrow w_{ji} + \Delta w_{ji} \text{ where } \Delta w_{ji} = \eta \delta_j x_{ji} \]
Termination conditions

- fixed number of iterations
- error falls below some threshold
- error on a separate validation set falls below some threshold

important:
- too few iterations reduce error insufficiently
- too many iterations can lead to overfitting the data
Adding Momentum

- one way to avoid local minima in the error surface or flat regions
- make the weight update in the $n^{th}$ iteration depend on the update in the $(n-1)^{th}$ iteration

$$\Delta w_{ji}(n) = \eta \delta_j x_{ji} + \alpha \Delta w_{ji}(n - 1)$$

Note: $\Delta w_{ji}(n - 1)$ represents the cumulative updates for this weight in the complete last epoch.

$0 \leq \alpha \leq 1$
Representational Power

**boolean functions:**
- every boolean function can be represented by a two-layer network

**continuous functions:**
- every continuous function can be approximated with arbitrarily small error by a two-layer network (sigmoid units at the hidden layer and linear units at the output layer)

**arbitrary functions:**
- each arbitrary function can be approximated to arbitrary accuracy by a three-layer network
Inductive Bias

- every possible assignment of network weights represents a syntactically different hypothesis
  \[ H = \{ \vec{w} | \vec{w} \in \mathbb{R}^{(n+1)} \} \]

- inductive bias: smooth interpolation between data points
Illustrative Example - Face Recognition

task:

- classifying camera image of faces of various people
- images of 20 people were made, including approximately 32 different images per person
- image resolution $120 \times 128$ with each pixel described by a greyscale intensity between 0 and 255
- identifying the direction in which the persons are looking (i.e., left, right, up, ahead)
Illustrative Example - Design Choices

- **input encoding:**
  - Image encoded as a set of $30 \times 32$
  - Pixel intensitiy values ranging from 0 to 255 linearly scaled from 0 to 1
  ⇒ reduces the number of inputs and network weights
  ⇒ reduces computational demands

- **output encoding:**
  - Network must output one of four values indicating the face direction
  - 1-of-n output encoding: 1 output unit for each direction
  ⇒ more degrees of freedom
  ⇒ difference between highest and second-highest output can be used as a measure of classification confidence
network graph structure:

- BACKPROPAGATION works with any DAG of sigmoid units
- question of how many units and how to interconnect them
- using *standard design*: hidden layer and output layer where every unit in the hidden layer is connected with every unit in the output layer
  ⇒ 30 hidden units
  ⇒ test accuracy of 90%
Advanced Topics

- hidden layer representations
- alternative error functions
- recurrent networks
- dynamically modifying network structure
### Summary

- able to learn discrete-, real- and vector-valued target functions
- noise in the data is allowed
- perceptrons learn hyperplane decision surfaces (linear separability)
- multilayer networks even learn nonlinear decision surfaces
- **BACKPROPAGATION** works on arbitrary feed-forward networks and uses gradient-descent to minimize the squared error over the set of training examples
- an arbitrary function can be approximated to arbitrary accuracy by a three-layer network
- **Inductive Bias**: smooth interpolation between data points