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Cognitive Models for Number Series Induction Problems

An Approach to Determine the Complexity of Number Series

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Abstract

Number series are an interesting research field for human pattern recognition and also for artificial intelligence. Though we need to determine the complexity of number series to use them purposeful. Our approach involves the analysis of number series in IQ tests, mainly considering human behavior/reasoning, because it is the basis for the application in computer science, especially in artificial intelligence. Thus our focus is on the complexity of number series. There are different approaches to analyze the complexity. On the footage of the Bounded Kolmogorov Complexity and the Structural Information Theory, we want to provide a new approach. We assume that we can gain a deeper insight by investigating the elemental units, operators and the structure of a number series in terms of complexity.

Studying the complexity of numbers involves the consideration of the pairity, the number of digits an integer has and the decomposability. Precisely the prime factorization and the greatest common divisor as well as the least common multiple are taken into account. Furthermore, we examine the operators between two numbers. It includes the elemental arithmetic operators and also relational operators defined on integers. Lastly, the resulting structure will be regarded. Based on the integer and operator analysis we can get an impression of the complexity of the structure. The aim will be to offer a precise study of number series as a basis for a complexity scale of number series.

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Chapter 1

Introduction

Solving number series in IQ tests is a challenge for both humans and machines. Furthermore it is part of the research field of artificial intelligence since 1960. There are many approaches of teaching machines the abilities of pattern recognition, how to derive rules from these patterns and applying them to continue the given number series. The most known examples are IGOR, ANN, MagicHaskeller and Asolver. Determing the performance of a system it is important to analyze not only given solutions but also the complexity of the tested data. Yet, there is no standard criteria for the classification of the complexity of number series. With a predefined categorization it would be easier evaluate number series and their application e.g. for IQ test. Besides it would also be easier to create a set of number series for IQ test.

In this work we will analyze different ways of determing the complexity of information and test their suitability for number series. With focus on Kolmogorov's definition of complexity, we will also concentrate on Structural Information Theory and Bounded Kolmogorov complexity, which are already used to classify number series. However there are still some difficulties like the definition of small values and the application of these classifications to both, humans and machines in the same way. Furthermore there are important parts missing. Since it is not possible for us within this work to analyze every single aspect of a number series problem we will focus on the determination of

- the complexity of numbers
- the complexity of arithmetic operations
- the complexity of structure

The scope of this work is to find a structure which can be applied to distinguish the information value of number series.

In the second chapter we will outline the state of research in the field of solving number series problems, their psychological relevance, and information complexity which are the footage of our work. First we will analyze Kolmogorov's definition of complexity. Afterwards we will outline Structural Information Theory and Bounded Kolmogorov Complexity, which are already used to classify number series. In addition we will apply the Structural Information Theory and Bounded Kolmogorov complexity to a set of number series. In a next step we will show the advantages and disadvantages of these methods. Furthermore we will describe the differences in the significance of the complexity on humans and machines. In chapter three we will deduce rules for determing the complexity from the state of the art. Furthermore we will explain which parts of number series are selected and why. Afterwards, in chapter four, we will present our approach to classify number series and analyze it. The results of our approach are summarized in the last chapter and further research is introduced.

Chapter 2

Approaches of Complexity

Before we introduce our own approach to the complexity of number series problems, we want to present some important theories for this purpose. Number series problems are part of many Intelligence quotient (IQ) tests. According to Sanghi and Dowe (2003) IQ tests were standardized approaches, developed by humans to measure the intelligence of humans, although there is no definition for intelligence. Designed by Alfred Binet, these tests are ment to evaluate the mental age of a person. The value expresses the IQ, the difference between the mentally and chronological age. The IQ is calculated as follows:

$$\frac{chronological\,age}{mental\,age} * 100$$

There are three possible result:

result = 100 The test person has the same age chronologically and mentally

result > **100** The test person's mentally age is higher than his or her chronologically age

result < **100** The test person's mentally age is lower than his or her chronologically age

One part of these IQ test are number series problems. They test the abilities of humans in pattern recognition, deriving rules from these patterns and applying them to continue the number series. Number series problems are finite integer sequences. The task for test audiences is to complete the sequence with the right integer. There are infinite ways of solving number series but only one unique solution can be true. The test audience is instructed through an example to find the right pattern and to complete the series. For Example:

$$2, 5, 7, 10, 12, ?$$

Definition 2.1 Number Series [Hofmann (2012)]
A number series is defined as $f(n) = x$ with domain of definition $\mathbb{N}$ and co-domain $\mathbb{R}$. $n \in \mathbb{N}$ with n $\geq$ 1 defines the position within the series. The co-domain $\mathbb{R}$ consists in IQ tests of natural numbers, although other like decimal numbers are also possible.

As you already determined the following number of the example series has to be 15. But how do humans solve number series problems? All you need to complete a series in most IQ tests are four abilities:

- pattern detection
- elementary mathematical skills
- deductive reasoning
- and in some cases further background knowledge.

All these abilities are already taught in elementary school. Number series can follow mathematical, psychological or cultural rules. There are many approaches in artificial intelligence in solving number series problems. Some of them like Asolver by Strannegård (2013b) and IGOR2 by Hofmann et al. (2014) try to replicate human problem solving on machines. Human reasoning on pattern detection problems was first described by Simon and Kotovsky Simon (1963) and developed by Holzman Holzman (1983).These contain:

- Relation detection: A relation between examples is determined and a hypothesis about how the examples are linked is created.
- Discovery of periodicity: The assumed relation between examples is tested and potential breaks are discovered. If breaks appear the length of periods is identified where arithmetic operations recur.
- Completion of Pattern Description: The hypothesis is tested on the whole series. The applied rules should cover all relations between examples elsewise new relations are determined and tested.
- Extrapolation: In the last step the hypothesis is applied to generate the following number and complete the series.

It is important especially for IQ tests that number series problems have just one right answer. Otherwise results would not be assimilable. Rules between examples can be explicit or recursive Hofmann (2012). Explicit functions use the position n to calculate the next example. Recursive functions use the predecessor to complete the series. Many series can be defined by either one of the functions or both. For Example:

$$3, 5, 7, 9, 11, ?$$

This sequence can be defined through a recursive or a explicit function:

- recursive: $f(n) = (2 * n) + 1$
- explicit: $f(n) = (n - 1) + 2$

The next number of the series is 13 no matter which function you use. The easiest way to measure the level of difficulty of a series is to compare the results of test participants. Series solved by many test persons seem to be easier than series solved by less people. To measure the difficulty of a series scientifically, it is necessary to analyze its complexity.

2.1 Kolmogorov Complexity

The Kolmogorov complexity is a common way to describe information structures such as number series. A number series is a finite binary sequence. There may be infinite ways to describe a sequence. The Kolmogorov complexity is a universal measurement that defines the amount of an object independent from the used describing method. The aim is to find the shortest possible description of an object that is needed to successfully output or reproduce the object.

> **Definition 2.2 Kolmogorov Complexity** [Nannen (2003), Strannegård (2013a)]
> The Kolmogorov complexity $K(x)$ of an Object x is the length of the shortest program p which can output x on the Universal Turing Machine and then halts. If you compare two programs p the shorter is less complex.

Most strings are not compressible because they contain no regularities or rules they follow. Thus there is no shorter description but the sequence itself. Those sequences are called *random*.

> **Definition 2.3 Randomness** [Li (2008)]
> An infinite sequence $a_1,\ a_2,\ \ldots$ of 0′s and 1′s is a random sequence in the special meaning of collective if the following two conditions are satisfied:
>
> 1. Let f_n be the number of 1′s among the first n terms of the sequence. Then
>
> $$\lim_{n\to\infty} \frac{f_n}{n} = p,\ for\ some\ p,\ 0 < p < 1.$$
>
> 2. A place-selection rule is a partial function ϕ from the finite binary sequences to the values 0 and 1 with the purpose of selecting one after another those indices n for which $\phi\ (a_1\ a_2 \ldots a_n - 1) = 1$. We require (1), with the same limit p, also for every infinite subsequence
>
> $$a_n1\ a_n2...$$
>
> obtained from the sequence by some admissible place-selection rule.

The Kolmogorov Complexity is not computable and thus not suitable for automatic pattern detection. Thus it is not suitable for pattern detection that can be used to solve number series problems. But there are two other approaches describing and categorizing number series we want to introduce:

- Structural Information Theory and
- Bounded Kolmogorov Complexity

2.2 Structural Information Theory

Another method to describe a number series is the Structural Information Theory (SIT). This theory was introduced by Leeuwenberg in 1971. In general, the theory is based on the organization of visual cognition which means that patterns will be structured by minimum costs. To predict the structuring of a pattern, a formal description has to be developed. Assuming that patterns are composed by elemental perception entities which are a sequence of symbols, a traversal can be transferred from these patterns. The aim is to describe the regularities within these sequences by analyzing possible periodicities. Therefore the regularity for the identity of sequences also has to suit any subset of these sequences. There are three possible regularities which occur within a sequence and ‚are generally considered to be indisputably relevant in visual perception'[van der Helm (1991)].

First, we introduce the iteration rule which can be ‚represented by the encoding of C = kkk ... kk into a code which, [...], may be denoted by n * (k) '[van der Helm (1991)]. In other words, a sequence of n identical elements e can be described by it(n, e). Hence, the iteration rule can represent the equal identity of elements.

Furthermore, there is the symmetry rule [Mühlpfordt (1999)] that represents a sequence

$$e_1,\ e_2 \ldots,\ e_n\ p\ e_n\ \ldots\ e_2\ e_1$$

in which p can also be empty (then denoted by lambda) and is described by

$$sy(e_1,\ \ldots,\ e_n,\ p).$$

At least there are the two alternation rules [Mühlpfordt (1999)] which describes a sequence

$$x_1\ e\ x_2\ e\ \ldots\ x_n\ e \qquad by \qquad al(e,\ x_1,\ \ldots,\ x_n)$$

and a sequence

$$e\ x_1\ e\ x_2\ \ldots\ e\ x_n \qquad through \qquad ar(e,\ x_1,\ \ldots,\ x_n)$$

To choose suitable regularities the concept of accessibility is used. This concept ‚implies that regularity and hierarchy in the code of a pattern corresponds directly to regularity and hierarchy in the pattern itself '[van der Helm (1991)]. The hierarchy can be described as follows. A hierarchical description implies that a description of a higher level is independent with regard to the description of the subjacent level. The regularities we already introduced above are satisfying the requirement of the hierarchy.

The occurring symbols in the descriptions of regularities should be recognized as diverse elements because they are not described by a regularity among them. If we abstract the concrete symbols within a description, the description equals a category of similar structured patterns, which nearly corresponds to the one-shot categorization [Mühlpfordt (1999)].

The structural information deriving from the structural description is minimal and is measured through the *information load*. This is equivalent to the number of different elements, i.e., a pattern with a length of n with k different symbols has a minimum information load of k and maximum of n. Due to the analysis of sequences of symbols the structural information theory has no domain specific operators like predecessor and successor. An algebraic extension was suggested by Dastani et al. 1997 [Mühlpfordt (1999)].

The advantage of the SIT is mainly the examination of the visual human pattern recognition. Especially the iteration and alternation rules can be used to describe a number series. The visual perception is then expressed in a mathematical way and therefore it should be applicable for machines, too.

2.3 Bounded Kolmogorov Complexity

In Strannegård (2013a) opinion, determining patterns in number series has subjective components. Different people can detect different patterns within the same series. Hence number series problems are not exclusively mathematical problems. The hypothesis a human creates for number series depends inter alia on his mathematical skills. There are infinite possible functions that can reproduce a series. Different functions can lead to ambiguity. Nevertheless in IQ tests only one solution is valid. IQ tests are originally created by humans and for human test audience. The complexity of series used in IQ tests is lower than for series in e.g. the Online Encyclopedia of Integer Sequences (OEIS) Strannegård (2013b). The background knowledge needed for solving series in OEIS is larger and includes higher mathematical skills. According to Strannegård (2013b) we have to understand human reasoning in number series problems because IQ test are created by humans and serve the purpose to measure first and foremost human intelligence.

The computational model by Strannegård (2013a) analyzes a number series in following categories:

- term describing a number series
- length of this term
- Distinction between complete or partial description
- using position n or not using n
- predicting small vs. large values

Definition 2.4 Term [Strannegård (2013a)]
Terms are: Numerals (e.g. 0,1,2) , the variable n, $f(n-c)$ with $c > 0$. If t_1 and t_2 are terms, then so is $(t_1 \star t_2)$ where $\star$ is one of the arithmetic operations $+, -, *$ or $/$

This distinction between terms is important for the second step in this analysis. The length of the term is determined in the following way:

Definition 2.5 Term Length [Strannegård (2013a)]
The length $|t|$ of a term t is defined by:
$|c|$ is the number of digits in c, disregarding trailing zeroes, when c is a numeral. The position $|n| = 1$.
Predecessors are counted as $|f(n-c)| = 1$
If t_1 and t_2 are terms, then $|(t_1 * t_2)| = |t_1| + |t_2| + 1$

To simulate human's limited working memory the length of each term was restricted to 8. Let's assume we want to analyze following series:

Number Series: 3 5 7 9 11 ?

There are infinite possible solutions, that can reproduce this sequence. Here are two of them:

- Solution 1: $f(2n + 1)$
- Solution 2: $f(n - 2) + 2$

The position of a number within the series is represented by n, $f(n - c)$ represents the number at position $n - c$. Following the rules of computing the length of a term we come to this conclusion: Solution 1 has the length 4, solution 2 has the length 3. Both are complete because they are not alternating. Solution 1 uses n, solution 2 does not. Both solutions use small values. Strannegård (2013a) decided to choose the most simple solution of a set of options. To evaluate the complexity they defined following rules:

- shorter terms preferred over longer terms
- simple descriptions preferred over descriptions modulo 2 etc.
- complete descriptions preferred over partial descriptions
- terms not including n (except as $f(n - c)$) preferred over terms containing n
- terms predicting smaller values preferred over others

Following these rules we conclude that solution 1 $f(2n + 1)$ contains more information than solution 2 $f(n - 2) + 2$. Thus solution 1 is more complex than solution 2.

The bounded Kolmogorov complexity proposed by Strannegård (2013a) derives information about number series. In comparison to SIT it takes alternating series into account. Furthermore it draws attention to the index and specific operators like predecessor and successor. It structures series on the basis of human reasoning. It is a good approach in rating the complexity of number series. Despite there are still some parts missing. Due to the fact that IQ tests are created for humans, the complexity of the used operators is not taken into account. Information deriving from the length of a term is minimal. Which length of a term is complex or simple? There is also no definition of small or large values.

Chapter 3

Complexity of the Basic Elements in Number Series

Our approach relative to the analysis of number series problems with respect to complexity involves different background knowledge. Before we apply the Kolmogorov Complexity, the Bounded Kolmogorov Complexity and the Structural Information Theory (SIT) to these number series problems, we would like to review some basic questions about number series. Which numbers and mathematical operations are used? How many numbers are given and how many should be given? Which size do the used constants have? Which structure arises from the combination of numbers and operators? Who will be tested by number series and therefore which background knowledge is relevant?

3.1 Numbers

The basic elements in number series are the numbers itself. We assume, that in number series problems the numbers are part of the integers $\mathbb{Z}$. Consequently we have a restriction to the complexity of integer values. Integer values can be positive or negative as well as odd or even. In number series we commonly find the positives $\mathbb{Z}^+$, especially in IQ tests. The proportion of odd and even numbers is (presumably) similar.

Definition 3.1 Parity [O'Regan (2013)]
The parity of an integer refers to whether the integer is odd or even. An integer n is odd if there is a remainder of 1 when it is divided by 2, and it is of the form $n = 2k + 1$. Otherwise, the number is even and of the form $n = 2k$. The sum of two numbers is even if both are even or both are odd. The product of two numbers is even if at least one of the numbers is even.

3.1.1 Decomposability

Therefore we want to begin with the complexity of positive integers. The positive integer values contain all numbers from one to infinity. But if we take a look at number series we can locate the positive integers of the given values between one and 100. The complexity of such integers can be ascertained by different ways. We like to present the prime factorization, greatest common factor and least common multiple from classical number theory as approaches to detect the complexity. Prime factorization respectively the Fundamental Theorem of Arithmetic means that every natural number $n > 1$ can be represented as the product of primes with

$$n = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$$

where p_i are prime numbers and a_i are positive integers for $i = 1..n$. Further, other than the order of the prime numbers, this factorization is unique. The necessary background knowledge for this purpose is the information about what a prime number is. Prime numbers and prime factorization are taught at school, thus the knowledge exists in humans particularly with a graduate from school. How about machines, do they know what a prime number is and do they need to know it? We expect that the knowledge must be transferred. One approach is to program the primality test.

Primality Test by Trial Divisions [Yan (2009)]

$$Test(p_i) \stackrel{def}{=} p_1, p_2, \cdots, p_k \leq \lfloor\sqrt{n}\rfloor, p_i \nmid n$$

Thus, if n passes Test(p_i), then n is prime:

$$n\ passes\ Test(p_i) \Rightarrow n \in Primes.$$

This test is applicable for "small" numbers but not for "big" numbers. The reason is that it needs $O(2^{\log(n)/2})$ bit operations. It might be a good variant for the IQ Tests where the integers in a number series are mostly between one and 100. If you want to test an integer n whether it is prime or not, you test the primes

$$\lfloor\sqrt{n}\rfloor = m,\ m \in primes.$$

Adjacent you test all the primes until m by trial divisions like: $\frac{n}{m}$. If none of the divisions returns a zero remainder, n is a prime number. If any test returns a zero remainder you can stop testing and infer that n is a composite.

The prime factorization is the basis for greatest common factor and least common multiple.

Definition 3.2 Greatest Common Divisor [Scheinerman (2006)]
Let a, b ∈ Z. We call an integer the greatest common divisor of a ∈ b, d is a common divisor of a and b. Additionally, if e is a common divisor of a and b, then e ∈ d. The greatest common divisor of a and b is denoted gcd(a,b).

Definition 3.3 Least Common Multiple [Scheinerman (2006)]
Let a and b integers. A common multiple of a and b is an integer n for

which a | n and b | n. We call an integer m the least common multiple of n provided m is positive, m is a common multiple of a and b, and if n is any other positive common multiple of a and b, then n ∈ m. The notation for the least common multiple of a and b is lcm(a,b).

The number of digits a series has is also interesting. As we mentioned before, in Strannegård (2013a) the number of digits in a term is analyzed. We apply this counting also to the integers in the series. The more digits a integer has, the more working memory is needed to handle it. This may cause less difficulties within artificial systems but challenges humans. As we mentioned before the human working memory is limited. These working memory restrictions can lead to difficulties with handling large numbers. We assume the complexity of numbers increases with the increasing number of digits and differences between them:

Example 1: 1 & 3
Both numbers are one-digit-numbers. The difference between these numbers can be detected easily. The possible patterns that can be applied to find a connection between those are also limited.

Example 2: 11 & 15
Both numbers are two-digit-numbers. The difference between both can be found in the ones-place.

Example 3: 31 & 17
Both integers are two-digit-integers. The difference between both can be found in the ones-place as well as in the tens-place. The complexity of these integers increased in comparison with the first example in two domains: number of digits and number of differences.

Example 4: 100 & 10
Both integers have different numbers of digits. Furthermore the same integers in the same order are used in both, *1* and *0*.

Example 5: 123 & 67
Both integers have different numbers of digits. There is also a difference in every place.

We assume that the complexity of a structure increases by adding more integers. The number of given integers in a series has to be at least three examples for each equation: two examples to create a hypotheses and the third to test and validate it. The number of given integers in our test cases was seven.

3.2 Operators

Operations are applied on two integers in the space of a number series. Common operators we found in number series are addition, subtraction and multiplication and division. However integer division is the most seldom within the four elementary operators. Further possible operations are exponentiation or special number series like the Fibonacci series.

Example (exponentiation): 1 8 27 64 125

$$f(n) = n^3$$

Example (Fibonacci): 1 1 2 3 5 8

$$f(n) = f(n-1) + f(n-2)\ n \geq 3$$

But, how do we describe the complexity of operators? First of all multiplication is repeated addition. Let's look e.g. at the sequence $2+2+2+2+2+2+2$. According to the Kolmogorov complexity the multiplication $7 * 2$ provides a shorter description and transfers the same information. Secondly, repeated multiplication is called exponentiation: $2*2*2*2*2*2*2$ is summarized as 2^7.

Furthermore addition and multiplication of integers lead to an increasing values of the series. Subtraction and division lead to a decreasing series. We suppose that the operator itself has almost no complexity. It is a very subjective impression whether someone perceives an operator as complex. Hence we like to propose that the combination of operators and integers is responsible for the complexity.

3.2.1 Addition

Humans do many computations in their heads. According to Bassarear (2011) addition is defined as: If A and B are disjoint sets, with a and b elements than the sum of a and b is equal to the number of elements in the union of the two sets:

$$a + b = n(A \cup B)$$

Furthermore Bassarear (2011) suggests different strategies for the addition of two-digit-numbers or more:

Leading digit: 39 + 57 = (30 + 50) + (9 + 7)
Some people prefer analyzing numbers from the *front*. They divide the integers into tens- and ones-places and calculate their values separately.

Compensation: 39 + 57 = 40 + 56
The ones-places are rounded up or down to the next 5 or 0.

Break and Bridge: 39 + 57 = (39 + 50) + 7
The second number is broken apart. The missing ones-places are added at the end of the calculation.

Compatible numbers: 39 + 57 = (35 + 55) + (4 + 2)
Search for digits whose addition results in "10".

Addition is more than just summarizing objects. If you want to use addition in base ten efficiently, knowledge about taking numbers apart is indispensable. Also the commutative and associative properties hold for addition.

Commutative property: $a + b = b + a$ for all a, b

Associative property: $a + (b + c) = (a + b) + c$ for all a, b, c

Addition is the first arithmetic operation taught in school and is of fundamental importance in everyday life. Thus we want to assume that it is deep-rooted in humans' minds.

3.2.2 Subtraction

If we define addition as adding two parts to create a whole, then subtraction can be seen as a whole and a part, while we need to find the other one: $c - a$ is equal to $a + ? = c$ There are two main ways to calculate subtractions by *adding up* and *subtracting down* Bassarear (2011). It is also possible to add the inverted value of the second number instead of using subtraction:

$$a - b = a + (-b)$$

. Furthermore subtraction can be divided into two sections:

- *take-away* and
- *comparison* problems

Take-away problems deal with sets where some parts are, as the name already says, taken away. In number series we deal with comparison problems, as two integers of the set are being compared. Furthermore the commutative and associative properties, we introduced in the section about addition, do not hold for subtraction.

3.2.3 Multiplication

As we mentioned before, multiplication can be seen as repeated addition. But multiplication is more than that. The general model of multiplication shows that the product is built from parts that share the same size or amount Bassarear (2011):

$$a * b = \underbrace{b + \cdots + b}_{a}$$

Distributive multiplication is used over addition and subtraction:

$$a\,(b + c) = ab + ac$$

Children learn multiplication with patterns. As we mentioned in the section numbers, there are several patterns that can be discovered: the product of two numbers is always even, when at least one factor is even. Only 1/4 of multiplication facts is an odd number Bassarear (2011). In comparison to addition a change of units is possible: 100 km/h * 2 h = 200 km. Multiplication can be also used to solve combinatorial analysis problems. Let us assume we have two different sets, one with n elements and an other with m elements. If we want to determine the number of possible combinations, repeated addition is not a appropriate solution. The representation of this problem leads to a *Cartesian product*:

> **Definition 3.4 Multiplication Principle** [Scheinerman (2006)]
> Consider two-element lists for which there are n choices for the first element, and for each element there are m choices for the second element. Then the number of such list is nm.

Moreover multiplication can be used to calculate enclosed areas. The commutative and associative properties hold for both, multiplication and addition.

Commutative property: $a * b = b * a$ for all a, b

Associative property: $a * (b * c) = (a * b) * c$ for all a, b, c

3.2.4 Division

Division is the last of the four fundamental operations taught in school. It can be seen as repeated subtraction but it is defined as Scheinerman (2006):

- a/b means there is a integer c with $b = a * c$
- $a/1 = a$ for all numbers.

Our definition of division implies that a division with 0 cannot be defined. The commutative and associative properties do not hold for division. Although it can be also seen as a multiplication with a reciprocal value:

$$a : b = a * \frac{1}{b}$$

There is also a definition for the division algorithm:

> **Definition 3.5 Division** [Scheinerman (2006)]
> Let $a, b \in \mathbb{Z}$ with $b > 0$. There exists integers q and r so that $a = qb + r$ and $0 = qb + r$.
> Moreover, there is only one such pair of integers (q, r) that satisfies these conditions. The integer q is called quotient and the integer r is called the remainder.

In other words, for any two whole numbers a, b with $b \neq 0$ there are two whole numbers q, r with $r \geq 0$ and $r < b$ that make the equation $a = qb + r$ true.

> **Example**
> Let $a = 19$ and $b = 5$. The quotient is $q = 3$ and the reminder is $r = 4$ so that $19 = 3 * 5 + 4$ and $0 \leq 4 \leq 5$.

3.2.5 Relations between arithmetic operators

We have discovered many similarities between the four fundamental arithmetic operations. Addition and multiplication lead to an increasing value. Subtraction and division lead to a decreasing value. Althought it is much more, multiplication can be seen as repeated addition and division can be seen as repeated subtraction. Addition is inverse subtraction. Multiplication is also a inverse division. There is a direct correlation between different properties like commutation and the operator. Moreover the hierarchy between the operators is regulated: multiplication and division have priority over addition and subtraction. To understand the underlying algorithms it is important to be able to apply the properties of the operations. Patterns provide a deeper understanding of the operators.

Multiplication and division are more abstract and hence, in our opinion, more complex operators. Many algorithms for handling basic mathematical operations have been taught during time to handle mathematical problems more efficiently. There is not *the* algorithm but many different. After all we still suppose that it is a very subjective impression whether someone perceives an operator as more or less complex.

3.3 Structure

Initially we focus on one function resulting from one operator and two numbers. Possible functions include at least one of the operators addition, subtraction, multiplication and division. The two integers can be identical or different, with a simple or a complex relationship. A variation of different integers with operators can lead to a diverse perception of the complexity.

Example 1: $f(n) = f(n-1) * 1$ $\quad$ ($Number\ Series:$ 1 1 1 1 1 1)
The function has the operation multiplication and every pair of integers are one, where the first integer derives from the number series itself.

Example 2: $f(n) = f(n-1) + 2$ $\quad$ ($Number\ Series:$ 4 6 8 10 12 14)
The function has the operation addition and every pair of integers are different but have a simple relationship to another, the greatest common divisor 2.

Example 3: $f(n) = f(n-1) - 4$ $\quad$ ($Number\ Series:$ 27 23 19 15 11 7)
The function has the operation subtraction and every pair of integers are different and have a complex relationship, since every pair has at least one prime number: (27,**23**), (**23**,**19**), (**19**,15), (15,**11**), (**11**,**7**).

In number series problems there are functions to get from one to another integer. Number series can be described by a single function or more. Different cases can be discovered. In the first case, we have a linear function as description:

Number Series 1: 1 2 3 4 5 6 $\quad$ ($f(n) = f(n-1) + 1$)

Number Series 2: 19 17 15 13 11 9 $\quad$ ($f(n) = f(n-1) - 2$)

Number Series 3: 3 6 12 24 48 96 $\quad$ ($f(n) = f(n-1) * 2$)

Number Series 4: 256 128 64 32 16 8 $\quad (f(n) = f(n-1)/2)$

The second case contains two functions with the same operation and different integers describing the number series.

Number Series 5: 8 12 10 16 12 $\quad (f_1(n) = f(n-2) + 4, f_2(n) = f(n-2) + 2)$

The last case contains two functions with different operations and optional different or identical integers.

Number Series 6: 2 7 2 7 2 7 $\quad (f_1(n) = f(n-1) + 5,\ f_2(n) = f(n-1) - 5)$

Number Series 7: 10 45 15 38 20 $\quad (f_1(n) = f(n-2) + 5,\ f_2(n) = f(n-2) - 7)$

The resulting structure depends on the given integers and the applied function/s. Hence, we can define various number series. First of all there are two basic definitions for series: arithmetic and geometric [White (2002); Bassarear (2011)].

Definition 3.6 Arithmetic Number Series
Arithmetic series are created by adding a constant. The relationship between two examples is defined through a constant. Thus all elements in the series share the same difference to their neighbor. The recursion for a arithmetic sequence a_n with common difference d is $a_n = a_1 + d$.

Definition 3.7 Geometric Number Series
Geometric series are based on multiplication of the same amount. All elements share the same ratio to their neighbor. The recursion for a geometric sequence g_n with common ratio r is: $g_n = r * g_n - 1$.

Series can be arithmetic or geometric. They can also be both or none of them. Alternating series for example can be both arithmetic and geometric:

1 2 4 5 10 11 22 23 46 $\quad -> \quad (n-1) + 1, f(n-1) * 2$

Definition 3.8 Alternating Number Series
A number series is alternating if (Number Series 6) the values of a series are directly related to their predecessors and variantly uses two or more operators, and (Number Series 7) number series which uses different operators and are variantly based on their pre- or pre-predecessor. The number series associated with (Number Series 7) can be separated into two or more series.

Although alternating number series are considered in Bounded Kolmogrov Complexity we want to provide a closer look by including the complexity of operators and constants. We can distinguish between three types of alternating series:

- alternating operators and consistent values
- consistent operators and alternating values
- alternating operators and alternating values

Furthermore alternating series with two or more operators of type addition and / or subtraction can be summarized to a non-alternating series. There are also series, that follow non-mathematical rules like: 30 28 31 30 31 30 31 31 30 31 . . .. As you already determined the next number has to be 30. The series represents the amount of days each month has. To solve this series background knowledge about months in a Gregorian calendar is needed. Further mathematical knowledge is needed e.g. for Fibonacci or Bernoulli numbers.

After we presented both, Bounded Kolmogorov Complexity and SIT, we state that the approach of machines and humans is quite different with respect to solving number series. Hence we assume, that we can employ various characteristics of the aforesaid complexity approximations to be viewed. Machines do not use the visual recognition like humans. They are also able to solve functions with elementary operators and integers without an appreciable calculation period. But many artificial systems have difficulties solving alternating number series, even with "small" values.

The relationship between two or more integers has different characteristics. Obviously, we have the comparison of two objects as a relation. Relational operators defined on integers are greater than (>), less than (<) and equals (=). Humans will first recognize if a number series is *ascending*, *descending*, *constant* or *hybrid*. Hybrid means that a number series contains different relational operators. We assume that unless we notice the tendency of the number series we can apply functions.

Example (equal): $1 = 1 = 1 = 1 = 1 = 1$

Example (greater than): $128 > 64 > 32 > 16 > 8 > 4$

Example (less than): $5 < 8 < 11 < 14 < 17 < 20$

Example (hybrid): $4 < 7 > 5 < 9 > 6 < 11$

If we have identified the relationship and the tendency of the number series, we can try to calculate the next number with a specific function. Here we assume, that we test our first function by calculating one existing number out of at least two other existing numbers in the number series. Once we have a suitable function, we can try all the other existing numbers. If we are successful, we can solve the number series by applying the function, else we have to test another function gradually. The resulting structure of the number series will be our next topic.

3.3.1 Structure Analysis using Bounded Kolmogorov Complexity and SIT

With regard to chapter two where we introduced Bounded Kolmogorov Complexity and Leeuwenbergs Structural Information Theory, we like to present our review of some important aspects. It is necessary to determine a number. Strannegård used the terms *small* and *big* values, but we like to use the approach of Holzman et al. where numbers are defined *simple* or *complex* instead.

Definition 3.9 Complex Numbers [Li (2008)]
The complex numbers are the field $\mathbb{C}$ of numbers of the form $x + iy$,

where x and y are real numbers and i is the imaginary unit equal to the square root of $-1, \sqrt{-1}$. When a single letter $z = x + i\ y$ is used to denote a complex number, it is sometimes called an "affix." In component notation, $z = x + i\ y$ can be written (x, y). The field of complex numbers includes the field of real numbers as a subfield.

We agree that it is significant to use the term length because humans prefer shorter terms over longer terms. The assumption is that humans begin to calculate with simple functions. If there is no solution one might use another function, which could be more complex than the initial one. On the contrary we assume that machines use the first possible function which solves a number series. It will not matter if this description is a shorter or a longer term, the focus is directly finding a solution.

The completion of a description is also interesting. This is associated with the identification of relational operators between numbers. If we determine the alternation, we have to look at the individual numbers to detect the relation between them. It seems obvious that a dependency between a number and its pre- predecessor or rather succ- successor requires two functions to calculate further elements of the number series.

n as the index of a number in a sequence is more interesting for machines. Humans do not use the index at first sight to solve a number series. Therefore further knowledge is needed to use the index. At first you have to know what a index is and also it is necessary to know that the initial number has the index zero and not one.

The SIT is more interesting for humans. Visual cognition is a ability humans have, not machines. Hence, the iteration rule is quite interesting. This rule is applied initially by recognizing the number series at first sight. In the case that the relational operator between every two integers of a number series is equal, i.e., greater than >, less than < or equals =, we know that the series is constant, ascending or descending. If a number series like Example 2 (4 6 8 10 12 14) is given we perceive that there is a common constant to be added, the two. Therefore we know that a constant is iteratively added every time we calculate a next number of a number series. The symmetry rule is less relevant for number series problems relating to our test data from IQ tests. Indeed, the alternation rules are important. If we have a look at the previous number series from Example (hybrid): 4 7 5 9 6 11 we probably recognize at first the changing relational operators between the numbers. Thus we realize the alternation. It is obvious that we have to apply two functions. Furthermore, we can divide the alternating series into two sub-series. In every sub-series we can apply one of the already tested functions.

Example: 4 7 5 9 6 11

$$Functions:\ f_1(n) = f(n-2) + 1,\ f_2(n) = f(n-2) + 2$$

To exemplify the previous description of splitting the number series we get:

$$Number\ Series\ 1a:\ 4\ 5\ 6 \quad and \quad Number\ Series\ 1b:\ 7\ 9\ 11$$

Then we can use the functions for calculation:

$$Function\ for\ 1a:\ f_1(n) = f(n-1) + 1 \qquad Function\ for\ 1b:\ f_2(n) = f(n-1) + 2$$

Within the sub-series we must consider the new function argument. In our case it will change from two to one.

Now we apply the alternation rule of SIT in the following way:

$$Series\ 1a: \ 4\ e\ 5\ e\ 6 \quad by \quad al(e,\ 4,\ 5,\ 6,\ldots)$$

$$Series\ 1b: \ e\ 7\ e\ 9\ e\ 11 \quad by \quad ar(e,\ 7,\ 9,\ 11,\ldots)$$

To combine these rules to match the original series 4 7 5 9 6 11 we need to transfer both rules into one: $al((ar),\ 4,\ 5,\ 6,\ldots)$.

We have presented our approach in detail. Initiated with the perception of the smallest units in number series, the numbers itself, followed by the used operators. And at least the resulting structure presented through the relation of numbers and operators together. In the below chapter we want to describe our approach on the basis of an example number series.

Chapter 4

Application of Complexity to Number Series

Solving number series in IQ tests requires knowledge of elementary mathematical skills and integers from 0 up to 100. The complexity is depending on the amount of information. The impact of an odd or even starting number has on the $n - th$ element in a sequence depends on the rules they follow. But regularities in succession in even or odd numbers, as well as in $<$ and $>$ can help finding the right formula to complete the sequence.

4.1 Example of Application

To introduce our approach in practice, we like to give an example number series (Add1) Hofmann (2012). We examine a number series in the following pattern:

The first table shows the numbers of a series. Here we start with the number of digits an integer has. Then we prove the oddness/parity and at last the decomposability. This should exemplify the first impression of the smallest units a number series has. In the next step (Table 4.2) we consider the relation between every integer with its successor. This relation is the next level to analyze a number series. That includes the relational operator and further the greatest common divisor (gcd) and the least common multiple (lcm). Lastly we view the difference in digits. This might be unobvious for the visual cognition but it can be an indicator for whether we use a more simple or complex function to solve a number series. Finally, in table three we focus the resulting structure of the number series. The structure is composed of the used operators and numbers. The outcome of this is the monotonicity and the applied function. A function is a special relation:

> **Definition 4.1 Function** [Scheinerman (2006)]
> A relation f is called a function provided $(a, b) \in f$ and $(a, c) \in f$ imply $b = c$.

We applied Elements from Bounded Kolmogorov Complexity and SIT. We excluded the length of a term from Strannegård (2013a), but we also preferred shorter descriptions, and terms without n. In our approach we extended the analysis to the complexity of numbers and operators.

The following tables derive from the number series: 2 5 8 11 14 17 20

	1st No.	2nd No.	3rd No.	4th No.	5th No.	6th No.	7th No.
Series	2	5	8	11	14	17	20
number of digits	1	1	1	2	2	2	2
even	x		x		x		x
odd		x		x		x	
Prime Factorization	2	5	2*2*2	11	2*7	17	2*2*5

Table 4.1: Focus on Numbers

In the first step we can recognize the alternation of even and odd numbers and the changing digits of numbers after the third number. On the basis of the prime factorization we can identify that in this part of the sequence every second integer is a prime.

	1st & 2nd	2nd & 3rd	3rd & 4th	4th & 5th	5th & 6th	6th & 7th
Series	2 ; 5	5 ; 8	8 ; 11	11 ; 14	14 ; 17	17 ; 20
Relational operator	<	<	<	<	<	<
gcd	1	1	1	1	1	1
lcm	10	40	88	154	238	340
difference in digits	ones place	ones place	both	ones place	ones place	both

Table 4.2: Focus on Relation

Table 4.2 shows, that the relational operator between every pair of integers is less than, so it is consistent. The values of numbers in this series are constantly increasing. The fact that the greatest common divisor is always one and the least common multiple varies is due to the prime factorization of table 4.1. If the integer pairs always consists of a even and odd number, and in our case at least one prime, the relation of these is more complex.

	number series
	2; 5; 8; 11; 14; 17; 20
used operators	+
used constants	3
Function	f (n-1) + 3
Structure	arithmetic
Monotonicity	ascending

Table 4.3: Focus on Structure

At least we get an impression of the hole structure. The used operator and constant between every pair of integers are addition (+) and three. Therefore we can build the function $f(n-1)+3$. It is an arithmetic function because we iterative add something and consequential we have an ascending number series.

Obviously, this fact was clear since we assert in table 4.2 that the operational operator between the integers is less than.

We detected some interesting relationships between our criteria. Recognizing an constantly increasing or decreasing value can exclude two possible operations (e.g. division and subtraction in a increasing series) and help to find the correct equation faster in the test. We could also determine that a change in the relational operator is a significant sign for a alternating series. Repetition of a integer is also another indicator for a alternating series. Used multiplications and divisions can be detected in the changes within the prime factorization. Changes in the ones-places within numbers with two-digits or more are caused more likely by addition or subtraction.

We assume that this number series is an example for a less difficult number series, because we only got one operator and one constant to build the function. And also the relational operator and therefore the arithmetic structure and the monotonicity are consistent. This could be an basic approach to define a number series with respect to complexity.

Let us assume that every table represents a layer. The bottom layer is table 4.1, the mid layer is represented by table 4.2 and the top layer is table 4.3. So, if we have consistent interpretations of characteristics in more than one layer the complexity of the number series decreases. For example, the interpretation of the relational operator ($<$) is conform with the resulting function ($f(n-1)+3$). Indeed it is just one possible solution but we know, that humans try the simplest solution first to check whether a function can solve the number series problem or not.

Chapter 5

The Future of Complexity Analysis

5.1 Summary

In this work we have present important approaches to analyze the complexity of number series. Kolmogorov and Leeuwenberg established fundamental theories which are applicable to number series problems. The Kolmogorov Complexity focuses the information structure in a more mathematical way. But unfortunately it is not computable and does not involve pattern detection as a possible method to solve a number series. Instead, the Structural Information Theory of Leeuwenberg focuses the cognition of patterns. Therefore we have applied some aspects like the iteration rule or the alternation rule. They describe the cognition of humans with respect to number series problems in a mathematical way. We presented the important three rules for pattern recognition in sequences. The iteration rule, for repeating elements in sequences. Also the alternation rules, which are crucial for the detection of the alternating numbers in number series. The symmetry rule is basically important but for number series problems not usefull, but rather interesting for other sequences (e.g. sequences of letters in words).

The Bounded Kolmogorov Complexity by Strannegård is a computable variant of the Kolmogorov complexity. We have shown that the computational model has five categories to analyze a number series. The term describing a number series and associated with it the length of a term. The distinction between a complete or partial description, which is necessary to number series problems with alternating series. A main advantage then is that it checks whether the index n is used or not, what is important especially for the computability of the complexity by machines. And at least the prediction of small and large numbers. The terms small and large are more difficult to define, so we have chosen the terms simple and complex numbers by Holzman instead.

Assuming that the cognition of complexity begins with the elemental components of a number series problem, we introduced our approach. The complexity of numbers, operators and the resulting structure are responsible for the complexity of number series. We have pointed out, that there are significant distinctions inside the cognition of numbers and the combination of numbers and operators. Also it is quite different whether a machine has to solve a number series problem or a human. Hence, we presented one approach how humans recognize number series by examples. The examples present number series with different focal points. The first one contains only the numbers which are analyzed in different ways: the number of digits, oddness/pairity and the prime factorization. Secondly, the relation of every pair of numbers (n and n+1) in a series is focused, including the relational operator, greatest common divisor and least common multiple as well as the difference in digits. At last we consider the structure. Therefore we extract the used operators and constants to ascertain the function. Then we deduce the structure and monotonicity.

5.2 Further Work

The analysis of number series problems needs some more perspectives to be considered. The phenomenon of number sense and intuition is a further psychological aspect which should be focused. We expect that humans have different points of view with regard to the approach of number series. There might be various perceptions of numbers and operators. Whereas one can solve functions with multiplication with ease but another person has noticeable problems in applying multiplication. Even the perception of numbers might be quite different. One possible reason can be different subjective performances of the working memory. However, here we have to cope with the task of determining simple and complex numbers in contrast to small and big values. We have introduced the complexity of numbers. But is it sufficient to prove the complexity of numbers, is it necessary to define small and big values as well? And therefore, is there a border between small and big values? If the answer is yes, where is the border located? Our results should also be analyzed empirically. Beside these issues it is also required to take a detailed look at the behavior of machines relating to number series.

A scale of the complexity of number series should be constructed by examining a variety of classification criteria. On the basis of this scale it might be possible to rank number series. The ranking could be used to select number series for solving specific difficult tasks like intelligence tests for humans or machines.

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Appendix A

Application Tables

The following Number Series Examples are from Hofmann (2012)

Table A.1: Number Series with Subtraction Operator *Sub3*

	1st No.	2nd No.	3rd No.	4th No.	5th No.	6th No.	7th No.
Series	25	22	19	16	13	10	7
number of digits	2	2	2	2	2	2	1
even		x		x		x	
odd	x		x		x		x
Prime Factorization	5*5	2*11	19	2*2*2*2	13	2*5	7

	1st & 2nd	2nd & 3rd	3rd & 4th	4th & 5th	5th & 6th	6th & 7th
Series	25 ; 22	22 ; 19	19 ; 16	16 ; 13	13 ; 10	10 ; 7
Relational operator	>	>	>	>	>	>
gcd	1	1	1	1	1	1
lcm	550	418	304	208	130	70
difference in digits	ones place	both	ones place	ones place	ones place	both

	number series
	25; 22; 19; 16; 13; 10; 7
used operators	-
used constants	3
Function	f (n-1) - 3
Structure	arithmetic
Monotonicity	descending

Table A.2: Number Series with Multiplication Operator *Mult1*

	1st No.	2nd No.	3rd No.	4th No.	5th No.	6th No.	7th No.
Series	3	6	12	24	48	96	192
number of digits	1	1	2	2	2	2	3
even		x	x	x	x	x	x
odd	x						
Prime Factorization	3	2*3	2*2*3	2*2*2*3	2*2*2*2*3	$2^5 * 3$	$2^6 * 3$

	1st & 2nd	2nd & 3rd	3rd & 4th	4th & 5th	5th & 6th	6th & 7th
Series	3 ; 6	6 ; 12	12 ; 24	24 ; 48	48 ; 96	96 ; 192
Relational operator	<	<	<	<	<	<
gcd	3	6	12	24	48	96
lcm	6	12	24	48	96	192
difference in digits	ones place	both	both	both	both	100s/ones place

	number series
	3; 6; 12; 24; 48; 96; 192
used operators	*
used constants	2
Function	f (n-1) * 2
Structure	geometric
Monotonicity	ascending

Table A.3: Number Series with Alternation *Alter6*

	1st No.	2nd No.	3rd No.	4th No.	5th No.	6th No.	7th No.
Series	8	12	10	16	12	20	14
number of digits	1	2	2	2	2	2	2
even	x	x	x	x	x	x	x
odd							
Prime Factorization	2*2*2	2*2*3	2*5	2*2*2*2	2*2*3	2*2*5	2*7

	1st & 2nd	2nd & 3rd	3rd & 4th	4th & 5th	5th & 6th	6th & 7th
Series	8 ; 12	12 ; 10	10 ; 16	16 ; 12	12 ; 20	20 ; 14
Relational operator	<	>	<	>	<	>
gcd	4	2	2	4	4	2
lcm	24	60	80	48	60	140
difference in digits	both	ones place	ones place	ones place	both	both

	number series
	8; 12; 10; 16; 12; 20; 14
used operators	+
used constants	4 ; 2
Function	f (n-1) + 4, f (n-1) +2
Structure	alternating
Monotonicity	hybrid